An algorithm for constructing minimal degree diagram representations of finite semigroups

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What is computable with \( n \) states?

What is the minimal number of states required for a particular computation?

By computation we mean a semigroup.
# Semigroups

**ABSTRACT**

**Definition**

A *semigroup* is a set $S$ with an associative binary operation $S \times S \rightarrow S$.

**Example (Flip-flop monoid)**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
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<tbody>
<tr>
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<td>a</td>
<td>b</td>
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<tr>
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Semigroups

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</table>

TRANSFORMATION
Definition
A transformation semigroup $(X, S)$ is a set of states $X$ and a set $S$ of transformations $s : X \rightarrow X$ closed under function composition.

Example (Transformations)
Flip-flop, the 1-bit memory semigroup

So these are computational devices... $\approx$ automata

With transformation semigroups, we get all semigroups. (Cayley’s theorem)
Enumeration: subsemigroups of the full transformation semigroup

∅

[11], [12], [21], [22]

[11], [12], [22]

[11], [12]

[12], [22]

[12], [21]

[22]

[11]

[12]
Data flood

Number of subsemigroups of full transformation semigroups.

<table>
<thead>
<tr>
<th>$\mathcal{T}_0$</th>
<th>#subsemigroups</th>
<th>#conjugacy classes</th>
<th>#isomorphism classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{T}_1$</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\mathcal{T}_2$</td>
<td>10</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>$\mathcal{T}_3$</td>
<td>1 299</td>
<td>283</td>
<td>267</td>
</tr>
<tr>
<td>$\mathcal{T}_4$</td>
<td>3 161 965 550</td>
<td>132 069 776</td>
<td>131 852 491</td>
</tr>
</tbody>
</table>

After discounting the state-relabelling symmetries the database of degree 4 transformation semigroups is still around 9GB.
subsemigroups of $\mathcal{T}_4$
subsemigroups of $T_4$ of size $\geq 157$
Enumeration/classification gives a simple table-lookup algorithm for finding minimal degree representations.

For up to degree 4 transformation representations...
Diagram Semigroups – Typical Elements

\[ \in \mathcal{P} B_n, \]

\[ \in B_n, \]

\[ \in \mathcal{P} \mathcal{T}_n, \]

\[ \in \mathcal{I}_n^* \]
Diagram Semigroups – Typical Elements

$\in \mathcal{I}_n$, $\in \mathcal{B}_n$

$\in \mathcal{T}_n$, $\in \mathcal{T}L_n$

$\in \mathcal{S}_n$, $1_n$
Diagram Semigroup Land

\[ S \hookrightarrow D_n \text{ where } D \in \{ \mathcal{PB}, B, \mathcal{PT}, \mathcal{T}, S, \mathcal{I}, \mathcal{I}^*, \mathcal{P}, \mathcal{B}, \mathcal{TL} \} \]
Temperley-Lieb, Jones monoid

Catalan numbers, sequences of well-formed parentheses.

corresponds to \((()((())))()\)

Applications in Physics: statistical mechanics, percolation problem.
Direct construction of embedding

Example

\[
\begin{array}{c|cc}
\mathbb{Z}_2 & 1 & 2 \\
1 & 1 & 2 \\
2 & 2 & 1 \\
\end{array}
\quad \rightarrow
\quad \begin{array}{c|cccc}
\mathcal{T}_2 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 4 & 4 \\
2 & 1 & 2 & 3 & 4 \\
3 & 1 & 3 & 2 & 4 \\
4 & 1 & 4 & 1 & 4 \\
\end{array}
\]
Direct construction of embedding

Example

<table>
<thead>
<tr>
<th>$\mathbb{Z}_2$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

$\rightarrow$

<table>
<thead>
<tr>
<th>$\mathcal{T}_2$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>4</td>
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<tr>
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<td>2</td>
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<td>4</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

$1 \mapsto 2$, $2 \mapsto 3$
How to search for the embedding map?

- Partial solutions exist (just put zero for an element not in the table), thus backtrack search can be used.

- Types of elements: the *index-period* of $s$ the smallest values $m \geq 1$, $r \geq 1$ such that $s^{m+r} = s^m$. The semigroup analogue of the order of a group element.

Ask about the story of this algorithm!
SHOULD I ASK?
I'M LOCKED OUT,
AND TRYING TO GET
MY ROOMMATE TO LET ME IN.

FIRST I TRIED HER CELL PHONE,
BUT IT'S OFF.

THEN I TRIED IRC, BUT SHE'S NOT ONLINE.

I COULDN'T FIND ANYTHING TO THROW AT HER WINDOW,

SO I SSH'D INTO THE MAC MINI IN THE LIVING ROOM
AND GOT THE SPEECH SYNTH TO YELL TO HER FOR ME.

BUT I THINK I LEFT THE VOLUME WAY DOWN,
SO I'M READING THE OS X DOCS TO LEARN TO SET THE VOLUME VIA COMMAND LINE.

AH.
I TAKE IT THE DOORBELL DOESN'T WORK?
Minimum diagram representation degree

Definition

$$\mu_D(S) = \min\{n \mid S \hookrightarrow D_n\}$$

where $$D \in \{\mathcal{P}, \mathcal{B}, \mathcal{P}_T, \mathcal{T}, S, \mathcal{I}, \mathcal{I}^*, \mathcal{P}, \mathcal{B}, \text{TL}\}$$

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>5</td>
</tr>
</tbody>
</table>

$$\mu_T(S)$$ | $$\mu_P(S)$$ | $$\mu_B(S)$$ | $$\mu_{TL}(S)$$
---|---|---|---
3 | 3 | 5 | 5
$P_1 \leftrightarrow T_2$
$P_2 \leftrightarrow T_5$
$B_1 \cong T_1$
$B_2 \leftrightarrow T_3$
$TL_1 \cong T_1$
$TL_2 \leftrightarrow T_2$
$TL_3 \leftrightarrow T_4$
$P_1 \leftrightarrow B_2$
$n$-generated embeddings

$S \xrightarrow{n} T$ if $S$ embeds into a subsemigroup of $T$ that can be generated by $n$ elements

A simple algorithm:
1. Find all $n$-generated subsemigroups of $T$.
2. Filter this set by the property that size is $\geq |S|$.
3. Try constructing embeddings.
$P_2$, 3-generated
\[ \mathcal{P}_2 \xleftarrow{2} \mathcal{P}_3? \]

The conjecture was: NO.
The conjecture was: NO.

But, there are 4, err... 3 distinct ways of this embedding.

\[
\mathcal{P}_2 \xrightarrow{2} \mathcal{P}_3 ?
\]

<table>
<thead>
<tr>
<th>Generators of $\mathcal{P}_2$</th>
<th>Solution 1</th>
<th>Solution 2</th>
<th>Solution 3</th>
</tr>
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<tbody>
<tr>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td><img src="image2.png" alt="Diagram 2" /></td>
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<tr>
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</tr>
<tr>
<td><img src="image9.png" alt="Diagram 9" /></td>
<td><img src="image10.png" alt="Diagram 10" /></td>
<td><img src="image11.png" alt="Diagram 11" /></td>
<td><img src="image12.png" alt="Diagram 12" /></td>
</tr>
</tbody>
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Oooooh!!!!
$\mathcal{B}_3$, the Brauer monoid of degree 3
$\mathcal{B}_3 \leftrightarrow \mathcal{B}_5$? There are at most 21 distinct ways.

$\mathcal{B}_3 \leftrightarrow \mathcal{B}_4?$, $\mathcal{B}_4 \leftrightarrow \mathcal{B}_5$? Answers: NO. (as expected)
Summary

Good news. We have an algorithm for constructing minimal degree representations for relatively small semigroups.

Bad news. We have an algorithm for constructing minimal degree representations for relatively small semigroups.
<table>
<thead>
<tr>
<th>Links</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>GAP</td>
<td><a href="http://www.gap-system.org">www.gap-system.org</a></td>
</tr>
<tr>
<td>SEMIGROUPS</td>
<td>www-groups.mcs.st-andrews.ac.uk/~jamesm/semigroups.php</td>
</tr>
<tr>
<td>SUBSEMI</td>
<td>bitbucket.org/egri-nagy/subsemi</td>
</tr>
<tr>
<td>VIZ</td>
<td>bitbucket.org/james-d-mitchell/viz</td>
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Blog on computational semigroup theory:

compsemi.wordpress.com

Other software packages: GraphViz, Gnuplot, \TeX
Thank You!