On the transition of $\mathcal{P}$-positions of Wythoff’s game

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Wythoff’s game
Wythoff’s game

- Two piles of finitely many coins.
- Two players move alternately,
  (i) either removing any positive number of coins from **one** pile:
    \[(a, b) \rightarrow^{(i,0)} (a-i, b), \quad (a, b) \rightarrow^{(0,i)} (a, b-i)\]
  (ii) or removing the **same** positive number of coins from **both** piles:
    \[(a, b) \rightarrow^{(i,i)} (a-i, b-i)\]
- The game **ends** when the two piles are empty.
- The player making the last move **wins**.
Winning and losing positions

A position is either

- $\mathcal{N}$-position: the player about to move next can win, or
- $\mathcal{P}$-position: lose.

There is no draw outcome.

Facts:

- Any move from a $\mathcal{P}$ position leads to an $\mathcal{N}$ position.
- From any $\mathcal{N}$ position, there exists a move leading to a $\mathcal{P}$ position.
The $\mathcal{P}$-positions

$$\mathcal{P} = \{([\phi n], [\phi^2 n]) | n \geq 0\} = \{([\phi n], [\phi n] + n) | n \geq 0\}$$

where $\phi = \frac{\sqrt{5}+1}{2} = 1.6180\ldots$: (the Golden ratio).

**Some first $\mathcal{P}$-positions**

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_n$</td>
<td>(1,2)</td>
<td>(3,5)</td>
<td>(4,7)</td>
<td>(6,10)</td>
<td>(8,13)</td>
<td>(9,15)</td>
<td>(11,18)</td>
<td>...</td>
</tr>
</tbody>
</table>

Example $(9, 10) \rightarrow (3, 0) \rightarrow (6, 10), (11, 14) \rightarrow (7, 7) \rightarrow (4, 7)$. 
Distribution of $\mathcal{P}$-positions
Recursive algorithm

Set

\[
\begin{cases}
  a_n = \lfloor \phi n \rfloor, \\
  b_n = \lfloor \phi^2 n \rfloor = a_n + n.
\end{cases}
\]

\[\mathcal{P} = \{(a_n, b_n) | n \geq 0\}\]

Let \(S\) be a set of some nonnegative integers. Set

\[\text{mex}(S) = \text{the smallest nonnegative integer NOT in } S.\]

**Algorithm:**

- \(a_n = \text{mex}\{a_i, b_i | 0 \leq i < n\},\)
- \(b_n = a_n + n.\)
Complementary sequences

Recall

\[
\begin{align*}
  a_n &= \lfloor \phi n \rfloor, \\
  b_n &= \lfloor \phi^2 n \rfloor = a_n + n.
\end{align*}
\]

Set

\[
A = \{ a_n | n \geq 1 \}, \\
B = \{ b_n | n \geq 1 \}
\]

Then \( A \) and \( B \) are \textbf{complementary sequences}:

\[
\begin{align*}
A \cap B &= \emptyset, \\
A \cup B &= \text{positive integers}.
\end{align*}
\]
Wythoff’s game has been studied intensively. Many variants have been examined. A slight modification in the rule of moves would result a big change in $\mathcal{P}$-positions.
Do there exist variants of Wythoff whose $\mathcal{P}$-positions, except possibly some finite number of them, can be obtained by adding a fixed integer $k \geq 1$ to the $\mathcal{P}$-positions of Wythoff.

We after

$$\mathcal{P} = S \cup \{(a_n + k, b_n + k) | n \geq n_0\}$$

with some finite set $S$ of positions.
Our variant

For each $k \geq 0$, we introduce a game $W_k$: each move from $(a, b)$ with $a \leq b$ is one of the following types:

$(i1)$ $(a, b) \rightarrow (a - i, b)$ or
$(i2)$ $(a, b) \rightarrow (a, b - i)$ or
$(ii)$ $(a, b) \rightarrow (a - i, b - i)$ such that $a - i \geq k$. 
Wythoff’s game

Question

Our variant

$W_k$
A further variant

For $0 \leq l \leq k$, we introduce a game $\mathcal{W}_{l,k}$: each move is one of the following types:

$(i1)$ $(a, b) \rightarrow (a - i, b)$ or
$(i2)$ $(a, b) \rightarrow (a, b - i)$ or
$(ii)$ $(a, b) \rightarrow (a - i, b - j)$ such that $\min(a - i, n - j) \geq l$ and $\max(a - i, b - j) \geq k$. 
$W_{l,k}$
For both games $\mathcal{W}_k$ and $\mathcal{W}_{l,k}$, the $\mathcal{P}$-positions form the set

$$\{(i, i) | 0 \leq i < k\} \cup \{(a_n + k, b_n + k) | n \geq 0\}.$$
Distribution of $\mathcal{P}$-positions: $k = 15$