Maximally Symmetric p-groups

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Joint work with:

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Notation: \( p \) always a prime and \( p > 3 \).

\( G \) always a finite \( p \)-group.

\( \Phi(G) = \text{Frattini subgroup of } G \), i.e. the smallest normal subgroup with elementary abelian quotient.
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**Burnside's Basis Theorem**: \( G \) is \( d \)-generated if and only if \( G/\Phi(G) \) is.

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G/\Phi(G) \cong F_p^d
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**Question:** What linear group does \( \text{Aut}(G) \) induce on \( G/\Phi(G) \)?
More questions:

Given $H \leq \text{GL}(d, p)$, is there a $p$-group $G$ such that $\text{Aut}(G)$ induces $H$ on $G/\mathfrak{z}(G)$?
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1. Given $H \leq \text{GL}(d, p)$, is there a $p$-group $G$ such that $\text{Aut}(G)$ induces $H$ on $G/\Phi(G)$?

2. What is the minimal exponent of such $G$?
More questions:

- Given \( H \leq \text{GL}(d, p) \), is there a \( p \)-group \( G \) such that \( \text{Aut}(G) \) induces \( H \) on \( G/\Phi(G) \)?

- What is the minimal \( p \)-exponent, \( p \)-order, or nilpotency class of such \( G \)?

**Definition:** Let \( \varphi : \text{Aut}(G) \rightarrow \text{Aut}(G/\Phi(G)) \cong \text{GL}(d, p) \).

**Definition:** For \( H \leq \text{GL}(d, p) \), call a finite \( p \)-group \( G \) a \( \Phi \)-symmetric \( p \)-group if:

\[
\varphi(\text{Aut}(G)) = H.
\]
Some answers: Let $H \leq GL(d,p)$. 
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Theorem (Heineken, Liebeck):
There exists a $p$-group $G$ of exponent $p^2$ and nilpotency class two such that $H$ is the group induced on $G/2(G)$. $\cong |H|$-generated.
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**Theorem (Bryant, Kovács):** There exists a $H$-symmetric $p$-group.

No bound on nilpotency class or exponent.
Theorem (Helleloid, Martin): Let \( n \geq 2 \).

\[
\lim_{n \to \infty} \left( \frac{\text{proportion of } d\text{-generated } p\text{-groups of } p\text{-length } \leq n}{\text{with automorphism group a } p\text{-group}} \right) = 1.
\]
Problem: Given $H$ maximal in $GL(d,p)$, find a $H$-symmetric $p$-group of class $\leq 3$ (ish?), of exponent $p$, of small (ish?) order.
Where to look?
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For a group $X$, the lower exponent-$p$ central series is:

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\lambda_0(X) = X, \quad \lambda_{i+1}(X) = \left[\lambda_i(X), X\right] (\lambda_i(X))^p.
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\[ \lambda_0(X) = X, \quad \lambda_{i+1}(X) = [\lambda_i(X), X] \left( \lambda_i(X) \right)^p. \]

- $\lambda_1(X) = [X, X]^p = \Phi(X)$ for $X$ a $p$-group.
- $\overline{\lambda}_i(X) := \lambda_i(X) / \lambda_{i+1}(X)$ is an el. ab. $p$-group.
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Eg. $X = D_8$. $\lambda_1(X) = Z(X)$, $\overline{\lambda}_0(X) \cong C_2^2$,
$\lambda_2(X) = 1$, $\overline{\lambda}_1(X) \cong C_2$. 
Let $c, d \in \mathbb{N}$.

**Definition:** The exponent $p$ class $c$ $d$-generator $p$-covering group is:

$$\Gamma_{c,d} := \frac{F_d}{(F_d)^p} \cdot c(F_d).$$

($F_d =$ free group on $d$ generators.)
Let \( c, d \in \mathbb{N} \).

**Definition:** The **exponent \( p \) class \( c \) \( d \)-generator \( p \)-covering group** is:

\[
\Gamma_{c,d} := \frac{F_d}{(F_d)^p \lambda_c(F_d)}.
\]

(\( F_d = \) free group on \( d \) generators.)

\( \Gamma_{c,d} \) is a **finite \( d \)-generated \( p \)-group** of class \( c \) and **exponent** \( p \).
Theorem (O'Brien): If $G$ is a $d$-generated $p$-group of class $c$ and exponent $p$ then

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$$\Gamma = \Gamma_{c,d} \quad \cdots \quad \exists$$

$$\lambda_1(\Gamma) \quad \cdots$$

$$\lambda_2(\Gamma) \quad \cdots$$

$$\lambda_c(\Gamma) \quad \cdots$$

$$\lambda_0(\Gamma) = \text{Im}_P^d \quad \leftarrow \quad GL(d,p)$$

$$\lambda_1(\Gamma)$$

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Theorem (O’Brien): If $G$ is a $d$-generated $p$-group of class $c$ and exponent $p$ then

$$\Gamma_{c,d} \rightarrow G.$$ 

Observation: $\Gamma_{c,d}$ is a $\text{GL}(d, p)$-symmetric $p$-group.
Theorem (O'Brien): Let $\lambda_c(\Gamma) < \Pi < \lambda_{c-1}(\Gamma_{c,d})$, 

$$N = N_{\text{GL}_d(d,p)}(\Pi)$$ and set $G = \Gamma / \Pi$. Then $\Phi(\text{Aut}(G)) = N$, $G$ is of exponent $p$, $d$-generated and class $c$. 
Theorem (O'Brien): Let $\lambda_c(\Gamma) < \Pi < \lambda_{c-1}(\Gamma_{c,d})$, 

$N = N_{\mathbb{C}L(d,p)}(\Pi)$ and set $G = \Gamma/\Pi$. Then $\emptyset(\text{Aut}(G)) = N$, $G$ is of exponent $p$, $d$-generated and class $c$.

i.e. $G$ is a $N$-symmetric $p$-group.
Problem: Given $H$ maximal in $\text{GL}(d,p)$, find a $H$-symmetric $p$-group of class $c$ and of exponent $p$. 
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Idea: Find $\lambda_c(\Pi_{c,d}) < \Pi < \lambda_{c-1}(\Pi_{c,d})$ which is $H$-invariant but not $GL(d, p)$-invariant.
Decomposition of $\overline{\lambda}_i(\Gamma)$ as a $GL(d, \rho)$-module:

\[
\begin{align*}
\overline{\lambda}_0(\Gamma) &= V \\
\overline{\lambda}_1(\Gamma) &= V^{(1,1)} = \Lambda^2 V \\
\overline{\lambda}_2(\Gamma) &= V^{(2,1,0)} = \Lambda^2 V \otimes V / \Lambda^3 V \\
\overline{\lambda}_3(\Gamma) &= V^{(3,1,0,0)} \oplus V^{(2,1,1,0)} \\
\overline{\lambda}_4(\Gamma) &= \ldots
\end{align*}
\]
Structure /Action of $H$:

Theorem (Aschbacher): Let $H$ be a maximal subgroup of $\text{GL}(d,p)$ with $\text{SL}(d,p) \not\leq H$. Then $H$ belongs to one of 8 "geometric families" or $H/\text{Z}(H)$ is an almost simple group and $H$ acts abs. irred.
Structure / Action of $H$:

**Theorem (Aschbacher):** Let $H$ be a maximal subgroup of $GL(d, p)$ with $SL(d, p) \not\subset H$. Then $H$ belongs to one of 8 "geometric families" or $H/2(H)$ is an almost simple group and $H$ acts abs. irred.

Families: Put $V = \mathbb{F}_p^d$.

- $C_1$: Subspace stabilizers,
- $C_2$: $V = V_1 \oplus \cdots \oplus V_r$,
- $C_3$: Extension field subgroups,
- $C_4$: $V = V_1 \otimes V_2$,
- $C_5$: Subfield subgroups,
- $C_6$: Extraspecial normalizers,
- $C_7$: $V = V_1 \otimes \cdots \otimes V_r$,
- $C_8$: Classical groups.
Action of $H$ on $\Sigma_i(\Pi)$?

E.g. $H \in C$, so $H$ preserves a subspace $U$ of $V$. 
Action of $H$ on $\Sigma_1(\mathbb{P})$?

e.g. $H \in C\!\!\!\!L$, so $H$ preserves a subspace $U$ of $V$.

Consider $S := \langle uv \mid u \in U, v \in V \rangle \subset \Pi^2V$.

Clearly $S$ is $H$-invariant.
Action of $H$ on $\lambda_i(\Pi)$?

E.g. $H \in C$, so $H$ preserves a subspace $U$ of $V$.

Consider

$$S := \langle uv \mid u \in U, v \in V \rangle \subset \Lambda^2 V.$$  

Clearly $S$ is $H$-invariant.

If $\dim U < d - 1$, then $S \neq \Lambda^2 V$, so $\exists \Pi \subset \lambda_i(\Gamma_{2,d})$ with

$$N_{\text{GL}(d,p)}(\Pi) = H$$

and $G := \Gamma_{2,d} / \Pi$ is a $H$-symmetric $p$-group of class $2$ and exponent $p$. 
e.g. $H \in C_8$, with $H = \text{Sp}(d, p)$ preserving $\mathcal{B}$, an alternating form.
e.g. $H \in C_8$, with $H=Sp(d,p)$ preserving $\beta$, an alternating form.

Define $\mu : \Lambda^2 V \to \mathbb{F}_p$ by

$$\mu(u \wedge v) = \beta(u,v).$$

$\mu$ is surjective, commutes with action of $H$ and $\text{codim } \ker \mu = 1$. 
e.g. $H \in G_8$, with $H = \text{Sp}(d, p)$ preserving $\beta$, an alternating form.

Define $\mu: \Lambda^2 V \to F_p$ by

$$\mu(uv) = \beta(u, v).$$

$\mu$ is surjective, commutes with action of $H$ and $\text{codim Ker } \mu = 1$.

Put $T = \text{Ker } \mu$.

Then $G := \Gamma_{2, d}/T$ is a $H$-symmetric $p$-group of exponent $p$,

class 2 and order $p^{d+1}$. 
e.g. \( H \in C_8 \), with \( H = \text{Sp}(d, p) \) preserving \( \beta \), an alternating form.

Define \( \mu : \mathbb{F}_p^2 V \to \mathbb{F}_p \) by

\[
\mu(\nu \mu) = \beta(\nu, \mu).
\]

\( \mu \) is surjective, commutes with action of \( H \) and \( \text{codim ker } \mu = 1 \).

Put \( T = \ker \mu \).

Then \( G := \Gamma_{2, d}/T \) is a \( H \)-symmetric \( p \)-group of exponent \( p \),

class 2 and order \( p^{d+1} \).

\((G \cong p^{d+1}.)\)
e.g. $H \in C_8$ with $H = \text{GO}(d, p)$ preserving $\beta$ a symm. bilinear form. $H$ acts irreducibly on $\Lambda^2 V$...
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$H$ acts irreducibly on $\Lambda^2 V$...

Action of $H$ on $\Lambda^2 V \otimes V / \Lambda^3 V$?

Define $\mu: V \otimes V \otimes V \rightarrow V$ by

$$\mu(u \otimes v \otimes w) = \beta(u, v) w.$$
e.g. $H \in \text{G}_8$ with $H = \text{GO}(d, p)$ preserving $\beta$ a symm. bilinear form.

$H$ acts irreducibly on $\Lambda^2 V$...

Action of $H$ on $\Lambda^2 V \otimes V / \Lambda^3 V$?

Define $\mu : V \otimes V \otimes V \to V$ by

$$\mu(u \otimes v \otimes w) = \beta(u, v) w.$$  

$\mu$ is a $H$-module homomorphism.
e.g. $\mathbf{H} \in C_7$ with $V = U \otimes W$ and $H \cong \text{GL}(r, \mathbb{P}) \cap S_2$ ($r^2 = d$).

Again $H$ is irreducible on $\Lambda^2 V$. On $V^{(2,0)} = \Lambda^2 V \otimes V / \Lambda^3 V$?
e.g. $\text{HeC}_7$ with $V = U \otimes W$ and $H \cong \text{GL}(r, p) \cong S_2$ ($r^2 = d$).

Again $H$ is irreducible on $\Lambda^2 V$. On $V^{(2,1,0)} = \Lambda^2 V \otimes V / \Lambda^3 V$?

Define $\mu : \Lambda^2 V \otimes V \rightarrow U^{(2,1,0)} \otimes S^3 W \oplus S^3 U \otimes W^{(2,1,0)}$

by $\mu : (a \otimes d \otimes b \otimes e) \otimes \text{cof} \mapsto \overline{ab \otimes c \otimes d \otimes e \otimes f} + a \otimes b \otimes c \otimes d \otimes e \otimes f$
e.g. $\mathsf{H} \subset \mathcal{C}_7$ with $V = U \otimes W$ and $\mathsf{H} \simeq \mathsf{GL}(r, p) \lesssim S_2$ ($r^2 = d$).

Again $\mathsf{H}$ is irreducible on $\Lambda^2 V$. On $V^{(2,1,0)} = \Lambda^2 V \otimes V / \Lambda^3 V$?

Define $\mu : \Lambda^2 V \otimes V \rightarrow U^{(2,1,0)} \otimes S^3 W \oplus S^3 U \otimes W^{(2,1,0)}$

by $\mu : (a \otimes d \otimes b \otimes e) \otimes c \otimes f \rightarrow \alpha \beta \otimes c \otimes d \otimes e \otimes f + a \beta \otimes b \otimes c \otimes d \otimes e \otimes f$

where $\simeq : \Lambda^2 U \otimes U \rightarrow U^{(2,1,0)},$

$\sim : \Lambda^2 W \otimes W \rightarrow W^{(2,1,0)}.$

$\Lambda^3 V \leq \text{Ker } \mu.$
Theorem (Bamberg, Glasby, M., Niemeyer):

Let \( p > 3 \) be prime and let \( d \in \mathbb{N} \).

Let \( H \) be a maximal subgroup of \( GL(d, p) \) with \( SL(d, p) \notin H \).

Suppose \( H \leq C_i \) with \( i \in \{1, 2, 3, 4, 7, 8\} \) or \( (H/2(H))' = Alt(m) \).

Then there exists a \( H \)-symmetric \( p \)-group of exponent \( p \) and class \( c \), with \( c \) below.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( V )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{U'}{U} )</td>
<td>2 if ( d \neq 2 ), 3 if ( d = 2 ).</td>
</tr>
<tr>
<td>2</td>
<td>( V \oplus \cdots \oplus V_r )</td>
<td>2 if ( r &lt; d ), 3 if ( r = d &gt; 2 ), 4 if ( r = d = 2 ).</td>
</tr>
<tr>
<td>3</td>
<td>( (F_{pe})^r )</td>
<td>3 if ( e = d = 3 ), 2 otherwise</td>
</tr>
<tr>
<td>4</td>
<td>( U \times W )</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>( V \oplus \cdots \oplus V_r )</td>
<td>2 if ( r \neq 2 ), 3 if ( r = 2 ).</td>
</tr>
<tr>
<td>8</td>
<td>( (V, \beta) )</td>
<td>2 if ( \beta ) alt., 3 if ( \beta ) symm.</td>
</tr>
<tr>
<td>( Alt(m) )</td>
<td>&quot;Fully Deleted&quot;</td>
<td>3</td>
</tr>
</tbody>
</table>
Thank you!