Contraction groups in locally compact groups

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A **topological group** is a group that is also a topological space, such that \((x, y) \mapsto xy\) and \(x \mapsto x^{-1}\) are continuous.

A topological group \(G\) has a largest connected subgroup \(G_0\). The component group \(G/G_0\) is **totally disconnected**.

\(G\) is **locally compact** if there is a compact neighbourhood of 1.

**Theorem (van Dantzig)**

Let \(G\) be a totally disconnected, locally compact group. Then the compact open subgroups of \(G\) form a base of neighbourhoods of the identity.
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**Theorem (van Dantzig)**

Let \(G\) be a totally disconnected, locally compact group. Then the compact open subgroups of \(G\) form a base of neighbourhoods of the identity.
Let $G$ be a topological group and let $\alpha$ be an endomorphism of $G$. Define the **contraction group**:

$$\text{con}(\alpha) := \{ g \in G \mid \alpha^n(g) \to 1 \text{ as } n \to +\infty \}$$

For $g \in G$ we define $\text{con}(g) := \text{con}(\alpha)$ where $\alpha : x \mapsto gxg^{-1}$.

**Examples**

- Let $G = \mathbb{C}^n$ and let $\alpha$ be a diagonalisable matrix. Then $\text{con}(\alpha)$ is the direct sum of all eigenspaces with eigenvalue $|\lambda| < 1$.

- Let $G = \text{Sym}(\mathbb{Z})$ (equipped with the pointwise convergence topology), and let $g$ be the permutation $x \mapsto x + 1$. Then $\text{con}(g)$ consists of all permutations $h$ of $\mathbb{Z}$ such that $h(y) = y$ for all $y \ll 0$. 
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Contraction groups in locally compact groups
The contraction group is very important for understanding the dynamics of an automorphism.

**Proposition**

Let $G$ be a totally disconnected, locally compact group and let $\alpha$ be an automorphism of $G$. Then the following are equivalent:

(i) $\text{con}(\alpha) = \text{con}(\alpha^{-1}) = 1$;

(ii) Every neighbourhood of the identity in $G$ contains an $\alpha$-invariant neighbourhood;

(iii) Every neighbourhood of the identity in $G$ contains an $\alpha$-invariant compact open subgroup.
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In a connected locally compact group $G$, the contraction group of $g \in G$ is sensitive to perturbations of $g$; that is, given a sequence $g_n \to g$, there is no reason for $\text{con}(g)$ to be approximated by $\text{con}(g_n)$.

**Example**

Let $G = \text{SL}_2(\mathbb{R})$ and let $g_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Then $\text{con}(g_\lambda) = \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \}$ for all $0 < \lambda < 1$, but $\text{con}(g_1)$ is trivial.
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However, the situation is different for totally disconnected, locally compact (t.d.l.c.) groups.

Let $G$ be a t.d.l.c. group, let $g \in G$ and let $U$ be a compact open subgroup of $G$. Then $U$ is **tidy above** for $g$ if $U = U_+ U_-$ for some subgroups $U_+$ and $U_-$ such that $U_+ \leq gU_+ g^{-1}$ and $U_- \geq gU_- g^{-1}$.

**Proposition (Willis)**

Let $G$ be a t.d.l.c. group and let $g \in G$. Then every neighbourhood of the identity in $G$ contains a compact open subgroup $U$ that is tidy above for $g$. 
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**Proposition (Willis)**

Let $G$ be a t.d.l.c. group and let $g \in G$. Then every neighbourhood of the identity in $G$ contains a compact open subgroup $U$ that is tidy above for $g$. 
Theorem 1 (Caprace–R–Willis)
Let $G$ be a t.d.l.c. group and let $g \in G$. Let $U$ be an open compact subgroup of $G$ that is tidy above for $g$, and let $u \in U$. Then there exists $t \in U \cap \text{con}(g^{-1})$ such that

$$\text{con}(gu) = t\text{con}(g)t^{-1}.$$ 

Corollary
Let $G$ be a t.d.l.c. group and $S(G)$ be the space of closed subgroups of $G$ (equipped with the Chabauty topology). Then the function

$$G \to S(G); \quad g \mapsto \overline{\text{con}(g)}$$

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An important complication in topological group theory is that interesting kinds of subgroups are not necessarily closed. For example, the contraction group of an element is not closed in general, and a topological group can be topologically simple (that is, there is no proper non-trivial closed normal subgroup) without being simple as an abstract group. However, sometimes we can control non-closed subgroups using topological methods.

**Theorem 2 (Caprace–R–Willis)**

Let $G$ be a t.d.l.c. group, let $A$ be a subgroup of $G$ (not necessarily closed), and let $g \in A$. Then.

$$\text{con}(g) \leq A \iff \text{con}(g) \leq N_G(A).$$
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We define the **Tits core** of a t.d.l.c. group $G$ to be the group

$$G^\dagger = \langle \text{con}(g) \mid g \in G \rangle.$$ 

**Theorem 3 (Caprace–R–Willis)**

Let $G$ be a t.d.l.c. group. Let $A$ be a dense subgroup of $G$. Then $G^\dagger \leq A$ if and only if $G^\dagger$ normalises $A$. Consequently, every dense subnormal subgroup of $G$ contains $G^\dagger$.

**Corollary**

Let $G$ be a topologically simple t.d.l.c. group. Suppose $G^\dagger$ is non-trivial. Then $G^\dagger$ is simple as an abstract group.
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Let \( G \) be a topologically simple t.d.l.c. group. Suppose \( G^\dagger \) is non-trivial. Then \( G^\dagger \) is simple as an abstract group.
Question

Let $G$ be a t.d.l.c. group. Suppose that $G$ is generated by a compact subset, and $G$ has no non-trivial discrete quotient. Is $G^\dagger$ necessarily dense in $G$?

By a result of Caprace–Monod, the problem reduces to the case when $G$ is topologically simple.

By results of Caprace–R.–Willis, if $G$ is topologically simple, we can additionally assume the following condition: 

(*) Given a non-trivial subgroup $K$ of $G$ such that $N_G(K)$ is open, then $C_G(K) = 1$.

(‘Most’ known examples of compactly generated simple t.d.l.c. groups do not satisfy (*).)
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