Discriminator Varieties of Double-Heyting Algebras

Christopher Taylor

Supervised by Dr. Tomasz Kowalski and Emer. Prof. Brian Davey

Department of Mathematics and Statistics
La Trobe University

8th ANZMC 2014
Let $L$ be a bounded distributive lattice and let $x \in L$.

- The *pseudocomplement* of $x$, denoted $x^*$, is the largest element $z$ such that $z \land x = 0$. Equivalently,

$$x \land z = 0 \iff z \leq x^*$$

- A *Heyting algebra* is a bounded distributive lattice with an additional operation $\rightarrow$, known as the *relative pseudocomplement*, where $\rightarrow$ satisfies the following equivalence

$$x \land z \leq y \iff z \leq x \rightarrow y$$

- In a Heyting algebra, we can define $x^* := x \rightarrow 0$. 

---

Chris Taylor

Discriminator Varieties of Double-Heyting Algebras
Pseudocomplements

Let $L$ be a bounded distributive lattice and let $x \in L$.

- The *pseudocomplement* of $x$, denoted $x^*$, is the largest element $z$ such that $z \land x = 0$. Equivalently,

$$x \land z = 0 \iff z \leq x^*$$

- A *Heyting algebra* is a bounded distributive lattice with an additional operation $\rightarrow$, known as the *relative pseudocomplement*, where $\rightarrow$ satisfies the following equivalence

$$x \land z \leq y \iff z \leq x \rightarrow y$$

- In a Heyting algebra, we can define $x^* := x \rightarrow 0$. 
Let $L$ be a bounded distributive lattice and let $x \in L$.

- The **pseudocomplement** of $x$, denoted $x^*$, is the largest element $z$ such that $z \land x = 0$. Equivalently,

  $$x \land z = 0 \iff z \leq x^*$$

- A **Heyting algebra** is a bounded distributive lattice with an additional operation $\rightarrow$, known as the *relative pseudocomplement*, where $\rightarrow$ satisfies the following equivalence

  $$x \land z \leq y \iff z \leq x \rightarrow y$$

- In a Heyting algebra, we can define $x^* := x \rightarrow 0$. 
Recall, the operation \( \rightarrow \) satisfies the following equivalence

\[
x \land z \leq y \iff z \leq x \rightarrow y
\]

Alternatively, a Heyting algebra is an algebra \( \langle H, \lor, \land, \rightarrow, 0, 1 \rangle \) where

1. \( \langle H, \lor, \land, 0, 1 \rangle \) is a bounded distributive lattice
2. \( x \rightarrow x \approx 1 \)
3. \( x \land (x \rightarrow y) \approx x \land y \)
4. \( x \land (y \rightarrow z) \approx x \land [(x \land y) \rightarrow (x \land z)] \)
5. \( z \land [(x \land y) \rightarrow x] \approx z \)

Thus the class of Heyting algebras forms an equational class.
Heyting Algebras

- Recall, the operation $\rightarrow$ satisfies the following equivalence

\[ x \land z \leq y \iff z \leq x \rightarrow y \]

- Alternatively, a Heyting algebra is an algebra $\langle H, \lor, \land, \rightarrow, 0, 1 \rangle$ where
  1. $\langle H, \lor, \land, 0, 1 \rangle$ is a bounded distributive lattice
  2. $x \rightarrow x \approx 1$
  3. $x \land (x \rightarrow y) \approx x \land y$
  4. $x \land (y \rightarrow z) \approx x \land [(x \land y) \rightarrow (x \land z)]$
  5. $z \land [(x \land y) \rightarrow x] \approx z$

- Thus the class of Heyting algebras forms an equational class
Recall, the operation $\rightarrow$ satisfies the following equivalence

$$x \land z \leq y \iff z \leq x \rightarrow y$$

Alternatively, a Heyting algebra is an algebra $\langle H, \lor, \land, \rightarrow, 0, 1 \rangle$ where

1. $\langle H, \lor, \land, 0, 1 \rangle$ is a bounded distributive lattice
2. $x \rightarrow x \approx 1$
3. $x \land (x \rightarrow y) \approx x \land y$
4. $x \land (y \rightarrow z) \approx x \land [(x \land y) \rightarrow (x \land z)]$
5. $z \land [(x \land y) \rightarrow x] \approx z$

Thus the class of Heyting algebras forms an equational class
A *dual Heyting Algebra* is simply the dual of a Heyting algebra. The dual of $\rightarrow$ is written $\leftarrow$ and satisfies the following equivalence

$$x \lor z \geq y \iff z \geq y - x$$

We also define the *dual pseudocomplement*, $x^+$, to be the smallest element $z$ such that $x \lor z = 1$. Equivalently,

$$x \lor z = 1 \iff z \geq x^+$$

In a dual Heyting algebra, we can define $x^+ := 1 - x$. 
A dual Heyting Algebra is simply the dual of a Heyting algebra. The dual of $\to$ is written $\to^{-}$ and satisfies the following equivalence

$$x \lor z \geq y \iff z \geq y - x$$

We also define the dual pseudocomplement, $x^+$, to be the smallest element $z$ such that $x \lor z = 1$. Equivalently,

$$x \lor z = 1 \iff z \geq x^+$$

In a dual Heyting algebra, we can define $x^+ := 1 - x$. 
A dual Heyting Algebra is simply the dual of a Heyting algebra. The dual of $\rightarrow$ is written $\rightarrow$ and satisfies the following equivalence

$$x \lor z \geq y \iff z \geq y - x$$

We also define the dual pseudocomplement, $x^+$, to be the smallest element $z$ such that $x \lor z = 1$. Equivalently,

$$x \lor z = 1 \iff z \geq x^+$$

In a dual Heyting algebra, we can define $x^+ := 1 - x$. 

Chris Taylor

Discriminator Varieties of Double-Heyting Algebras
Double-Heyting algebras

An algebra \( \langle H, \lor, \land, \rightarrow, -, 0, 1 \rangle \) is a **double-Heyting algebra** if

- \( \langle H, \lor, \land, \rightarrow, 0, 1 \rangle \) is a **Heyting algebra**
- \( \langle H, \lor, \land, -, 0, 1 \rangle \) is a **dual Heyting algebra**
The Discriminator Term

- An algebra $A$ is called a *discriminator algebra* if it has a *discriminator term*, i.e. a term $t(x, y, z)$ where

\[
 t(x, y, z) = \begin{cases} 
 x & \text{if } x \neq y \\
 z & \text{otherwise}
\end{cases}
\]

- Example: finite fields of order $p$, we have

\[
 t(x, y, z) = z + (x - z)(y - x)^{p-1}
\]

- A *discriminator variety* is an equational class where there is a term $t$ that is a discriminator term on every subdirectly irreducible member of the class
An algebra $A$ is called a **discriminator algebra** if it has a **discriminator term**, i.e. a term $t(x, y, z)$ where

$$
t(x, y, z) = \begin{cases} 
  x & \text{if } x \neq y \\
  z & \text{otherwise}
\end{cases}
$$

Example: finite fields of order $p$, we have

$$
t(x, y, z) = z + (x - z)(y - x)^{p-1}
$$

A **discriminator variety** is an equational class where there is a term $t$ that is a discriminator term on every subdirectly irreducible member of the class.
An algebra $A$ is called a \textit{discriminator algebra} if it has a \textit{discriminator term}, i.e. a term $t(x, y, z)$ where

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise} \end{cases}$$

Example: finite fields of order $p$, we have

$$t(x, y, z) = z + (x - z)(y - x)^{p-1}$$

A \textit{discriminator variety} is an equational class where there is a term $t$ that is a discriminator term on every subdirectly irreducible member of the class.
Let $H$ be a double-Heyting algebra.

- Recall that the pseudocomplement of $x \in H$ is given by $x^* := x \to 0$
- Dually, the dual pseudocomplement of $x \in H$ is given by $x^+ := 1 - x$
- We set $x^{0(+*]} = x$, then define $x^{(n+1)(+*)} := (x^{n(+*)})^{+*}$

**Lemma**

*For any $x$ we have*

\[ x \geq x^{++} \geq x^{+++} \geq \cdots \geq x^{n(+*)} \geq \cdots \]
The $+$* operation

Let $H$ be a double-Heyting algebra.

- Recall that the \textit{pseudocomplement} of $x \in H$ is given by $x^* := x \rightarrow 0$
- Dually, the \textit{dual pseudocomplement} of $x \in H$ is given by $x^+ := 1 - x$
- We set $x^{0(+*)} = x$, then define $x^{(n+1)(+*)} := (x^{n(+*)})^{+*}$

\textbf{Lemma}

For any $x$ we have

$$x \geq x^{++} \geq x^{++*} \geq \cdots \geq x^{n(+*)} \geq \cdots$$
The $+^*$ operation

Let $H$ be a double-Heyting algebra.

- Recall that the pseudocomplement of $x \in H$ is given by $x^* := x \to 0$
- Dually, the dual pseudocomplement of $x \in H$ is given by $x^+ := 1 - x$
- We set $x^{0(+^*)} = x$, then define $x^{(n+1)(+^*)} := (x^{n(+^*)})^{+^*}$

**Lemma**

For any $x$ we have

\[x \geq x^{+^*} \geq x^{++^*} \geq \cdots \geq x^{n(+^*)} \geq \cdots\]
The $+^*$ operation

Let $H$ be a double-Heyting algebra.

- Recall that the pseudocomplement of $x \in H$ is given by $x^* := x \rightarrow 0$
- Dually, the dual pseudocomplement of $x \in H$ is given by $x^+ := 1 - x$
- We set $x^{0(+^*)} = x$, then define $x^{(n+1)(+^*)} := (x^{n(+^*)})^+$

**Lemma**

For any $x$ we have

$$x \geq x^{+^*} \geq x^{+^{+^*}} \geq \cdots \geq x^{n(+)^*} \geq \cdots$$
The \( \mathbf{+*} \) operation

Let \( H \) be a double-Heyting algebra.

- Recall that the \textit{pseudocomplement} of \( x \in H \) is given by \( x^* := x \rightarrow 0 \)
- Dually, the \textit{dual pseudocomplement} of \( x \in H \) is given by \( x^+ := 1 - x \)
- We set \( x^{0(\mathbf{+*})} = x \), then define \( x^{(n+1)(\mathbf{+*})} := (x^{n(\mathbf{+*})})^{\mathbf{+*}} \)

\[ x \geq x^+ \geq x^{++} \geq \cdots \geq x^{n(\mathbf{+*})} \geq \cdots \]
Let $H$ be a double-Heyting algebra.

- For a set $F \subseteq H$ we say $F$ is a filter if
  - $F$ is an up-set
  - $F$ is closed under the operation $\land$
- If $F$ is also closed under the term operation $\vdash^*$ then we say $F$ is a normal filter on $H$
- For any $x \in H$, the normal filter generated by $x$ is given by
  
  $$N(x) = \bigcup_{m \in \omega} x^m(\vdash^*)$$
Let $H$ be a double-Heyting algebra.

- For a set $F \subseteq H$ we say $F$ is a filter if
  - $F$ is an up-set
  - $F$ is closed under the operation $\land$

- If $F$ is also closed under the term operation $+\ast$ then we say $F$ is a normal filter on $H$

- For any $x \in H$, the normal filter generated by $x$ is given by

$$N(x) = \bigcup_{m \in \omega} \uparrow x^m(+\ast)$$
Let $H$ be a double-Heyting algebra.

- For a set $F \subseteq H$ we say $F$ is a filter if
  - $F$ is an up-set
  - $F$ is closed under the operation $\land$
- If $F$ is also closed under the term operation $\dag^*$ then we say $F$ is a normal filter on $H$
- For any $x \in H$, the normal filter generated by $x$ is given by

\[ N(x) = \bigcup_{m \in \omega} \uparrow x^m(\dag^*) \]
Let $H$ be a double-Heyting algebra.

- For a set $F \subseteq H$ we say $F$ is a filter if
  - $F$ is an up-set
  - $F$ is closed under the operation $\wedge$
- If $F$ is also closed under the term operation $+^\ast$ then we say $F$ is a normal filter on $H$
- For any $x \in H$, the normal filter generated by $x$ is given by

$$N(x) = \bigcup_{m \in \omega} \uparrow x^m(\ast^+)$$
Congruences are determined by normal filters

Let $\text{NF}(H)$ denote the lattice of normal filters of $H$

For any $F \in \text{NF}(H)$ define the congruence $\theta(F)$ by

$$(x, y) \in \theta(F) \text{ iff } x \land f = y \land f \text{ for some } f \in F$$

Theorem

*The map $\theta : \text{NF}(H) \rightarrow \text{Con}(H)$ as given above is an isomorphism.*
Congruences are determined by normal filters

- Let $\text{NF}(H)$ denote the lattice of normal filters of $H$.
- For any $F \in \text{NF}(H)$ define the congruence $\theta(F)$ by:
  
  $$(x, y) \in \theta(F) \text{ iff } x \land f = y \land f \text{ for some } f \in F$$

**Theorem**

The map $\theta : \text{NF}(H) \rightarrow \text{Con}(H)$ as given above is an isomorphism.
Let $\text{NF}(H)$ denote the lattice of normal filters of $H$

For any $F \in \text{NF}(H)$ define the congruence $\theta(F)$ by

$$(x, y) \in \theta(F) \text{ iff } x \land f = y \land f \text{ for some } f \in F$$

**Theorem**

*The map $\theta : \text{NF}(H) \rightarrow \text{Con}(H)$ as given above is an isomorphism.*
Lemma

Let $H$ be a double-Heyting algebra. If $H$ is simple, then for every $x \in H$ with $x \neq 1$ there exists some $n_x < \omega$ where $x^{n_x}(+) = 0$.

Proof.

If $H$ is simple there can only be two normal filters on $H$. In particular, for any $x \in H$ with $x \neq 1$, we have

$$N(x) = H$$

$$\iff 0 \in N(x)$$

$$\iff (\exists n_x < \omega) \ 0 \in x^{n_x}(+)$$

as $N(x) = \bigcup_{m \in \omega} x^m(+)$. 

Chris Taylor
Discriminator Varieties of Double-Heyting Algebras
The class $\mathcal{D}_n$

The class $\mathcal{D}_n$ is the equational class of double-Heyting algebras satisfying the following equation $H$

$$x^{(n+1)(+*)} = x^n(+)$$
The class $\mathcal{D}_n$

**Theorem**

$\mathcal{D}_n$ is a discriminator variety for every $n < \omega$

**Proof sketch.**

We omit the proof that if $H \in \mathcal{D}_n$ is subdirectly irreducible, then

$$x^{n(\ast\ast)} = \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{otherwise}
\end{cases}$$

Put $x \leftrightarrow y := (x \to y) \land (y \to x)$. The discriminator term is

$$[x \land (x \leftrightarrow y)^{n(\ast\ast)+}] \lor [z \land (x \leftrightarrow y)^{n(\ast\ast)}]$$
The main result

An equational class \( \mathcal{K} \) is said to be *semisimple* if every subdirectly irreducible algebra in \( \mathcal{K} \) is simple.

It is well-known that every discriminator variety is semisimple. In general, the converse is not true.

For double-Heyting algebras, it is true

**Theorem**

Let \( \mathcal{V} \) be an equational class of double-Heyting algebras. Then the following are equivalent.

1. \( \mathcal{V} \) is a discriminator variety
2. \( \mathcal{V} \) is semisimple
3. \( \mathcal{V} \subseteq \mathcal{D}_n \) for some \( n < \omega \)
The main result

- An equational class $\mathcal{K}$ is said to be *semisimple* if every subdirectly irreducible algebra in $\mathcal{K}$ is simple.

- It is well-known that every discriminator variety is semisimple. In general, the converse is not true.

- For double-Heyting algebras, it is true.

**Theorem**

Let $\mathcal{V}$ be an equational class of double-Heyting algebras. Then the following are equivalent.

1. $\mathcal{V}$ is a discriminator variety
2. $\mathcal{V}$ is semisimple
3. $\mathcal{V} \subseteq \mathcal{D}_n$ for some $n < \omega$
The main result

- An equational class $\mathcal{K}$ is said to be *semisimple* if every subdirectly irreducible algebra in $\mathcal{K}$ is simple.
- It is well-known that every discriminator variety is semisimple. In general, the converse is not true.
- For double-Heyting algebras, it is true

**Theorem**

Let $\mathcal{V}$ be an equational class of double-Heyting algebras. Then the following are equivalent.
1. $\mathcal{V}$ is a discriminator variety
2. $\mathcal{V}$ is semisimple
3. $\mathcal{V} \subseteq D_n$ for some $n < \omega$
The main result

- An equational class $\mathcal{K}$ is said to be *semisimple* if every subdirectly irreducible algebra in $\mathcal{K}$ is simple.
- It is well-known that every discriminator variety is semisimple. In general, the converse is not true.
- For double-Heyting algebras, it is true

**Theorem**

*Let $\mathcal{V}$ be an equational class of double-Heyting algebras. Then the following are equivalent.*

1. $\mathcal{V}$ is a discriminator variety
2. $\mathcal{V}$ is semisimple
3. $\mathcal{V} \subseteq \mathcal{D}_n$ for some $n < \omega$