Locally triangular graphs
and rectagraphs with symmetry

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Joint work with John Bamberg, Alice Devillers, and Cheryl Praeger
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\begin{center}
\begin{tikzpicture}

\node (u) at (0,0) [label=above:$u$] {};

\node (v1) at (1,1.732) {};
\node (v2) at (1,0) {};
\node (v3) at (0,1.732) {};
\node (v4) at (-1,0) {};
\node (v5) at (-1,1.732) {}

\draw[thick] (u) -- (v1) -- (v2) -- (v3) -- (v4) -- (v5) -- (u);
\draw[thick,blue] (v1) -- (v2) -- (v3) -- (v4) -- (v5)
\end{tikzpicture}
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A graph $\Gamma$ is \textbf{locally} $\Delta$ for some graph $\Delta$ if for every vertex $u \in V\Gamma$, the graph induced by the neighbourhood $\Gamma(u)$ is isomorphic to $\Delta$. 
A graph $\Gamma$ is **locally 2-arc transitive** if there exists $G \leq \text{Aut}(\Gamma)$ such that, for every $u \in V\Gamma$, the stabiliser $G_u$ acts transitively on the 2-arcs starting at $u$: 

![Triangle 2-geodesic](image)
A graph $\Gamma$ is **locally 2-arc transitive** if there exists $G \leq \text{Aut}(\Gamma)$ such that, for every $u \in V\Gamma$, the stabiliser $G_u$ acts transitively on the 2-arcs starting at $u$:

![Diagram showing a triangle and a 2-geodesic graph representing locally 2-arc transitive graphs.](image-url)
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![Diagram of a triangle and a 2-geodesic]

If $\Gamma$ is a connected non-complete graph with girth 3, then $\Gamma$ is **never** locally 2-arc transitive.
Suppose that for each $u \in V\Gamma$, there are two orbits of $\text{Aut}(\Gamma)_u$ on the 2-arcs starting at $u$, namely the triangles and the 2-geodesics.
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Thus the permutation group $\text{Aut}(\Gamma)^{\Gamma(u)}_u$ induced by $\text{Aut}(\Gamma)_u$ on $\Gamma(u)$ is transitive of rank 3 for all $u \in V\Gamma$. 
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Thus the permutation group \( \text{Aut}(\Gamma)^{\Gamma(u)}_u \) induced by \( \text{Aut}(\Gamma)_u \) on \( \Gamma(u) \) is transitive of rank 3 for all \( u \in V\Gamma \).

In fact, the converse holds for all connected non-complete graphs with girth 3.
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A graph $\Gamma$ is **locally rank 3 with respect to** $G$ if the following hold:

(i) $\Gamma$ has no vertices with valency 0,
(ii) $G \leq \text{Aut}(\Gamma)$,
(iii) For all $u \in V\Gamma$, the group $G_{\Gamma}^{(u)}$ is transitive of rank 3.

We also say that $\Gamma$ is locally rank 3 if it is locally rank 3 with respect to some $G$. 
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- The **triangular graph** $T_n$ for $n \geq 3$:
  - Vertex set: $\binom{n}{2} := \{\{i, j\} : 1 \leq i < j \leq n\}$.
  - Adjacency: $\{i, j\} \sim \{k, \ell\} \iff |\{i, j\} \cap \{k, \ell\}| = 1$. 
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Aut($T_n$) = Aut($\overline{T}_n$) is $S_n$ for $n \neq 4$ and $S_4 \times C_2$ for $n = 4$. 

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The complement $\overline{T}_5$ of $T_5$ is an example of a Petersen Graph.

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All of these graphs are locally rank 3.
A connected graph $\Gamma$ is locally rank 3 and locally $T_n$ if and only if $\Gamma$ is the halved graph of one of the following graphs.

(i) The $n$-cube $Q_n$ where $n \geq 3$.
(ii) The folded $n$-cube $\Box_n$ where $n$ is even and $n \geq 8$.
(iii) The bipartite double of the coset graph of the binary Golay code $C_{23}$.
(iv) The coset graph of the extended binary Golay code $C_{24}$. 
A **rectagraph** is a connected triangle-free graph in which any 2-arc lies in a unique quadrangle.
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**Lemma (Neumaier, 1985)**

A connected graph $\Gamma$ is locally $T_n$ if and only if $\Gamma$ is a halved graph of a bipartite rectagraph of valency $n$ with $c_3 = 3$. 
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Lemma (Neumaier, 1985)

A connected graph \( \Gamma \) is locally \( T_n \) if and only if \( \Gamma \) is a halved graph of a bipartite rectagraph of valency \( n \) with \( c_3 = 3 \).

Let \( G \) be a group acting on a set \( \Omega \) with \( |\Omega| \geq 4 \). We say that \( G \) is 4-homogeneous if \( G \) is transitive on the set of 4-subsets of \( \Omega \).
Theorem (Bamberg-Devillers-F.-Praeger, 2014)

Let $\Pi$ be a rectagraph with $c_3 = 3$ and no 5-cycles. There exists $u \in V\Pi$ such that $|\Pi(u)| \geq 4$ and $\text{Aut}(\Pi)u$ is 4-homogeneous on $\Pi(u)$ if and only if $\Pi$ is one of the following.

(i) The $n$-cube $Q_n$ where $n \geq 4$.

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Let $\Delta$ and $\Pi$ be graphs. A surjective map $\pi : \Delta \rightarrow \Pi$ is a covering if $\pi$ induces a bijection from $\Delta(x)$ onto $\Pi(x\pi)$ for all $x \in V\Delta$. 

Lemma Let $\Pi$ be a rectagraph of valency $n$ with $c_3 = 3$ and no 5-cycles.

(i) For any $u \in V\Pi$, there exists a covering $\pi : Q_n \rightarrow \Pi$ such that $0\pi = u$ (Brouwer-Cohen-Neumaier, 1989).

(ii) $\Pi$ is a normal quotient of $Q_n$ by $K = \{g \in \text{Aut}(Q_n) : g \circ \pi = \pi\}$ (Matsumoto, 1991).
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\[ \forall u, v \in V\Pi \]

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What next?

Consider locally Grassmann graphs:

Vertex set: The 2-subspaces of an $F^q$-vector space.

Adjacency: Two 2-subspaces are adjacent whenever their intersection has dimension one.

Examples of locally Grassmann graphs include:

The graph of alternating forms over $F^2$.

The graph of quadratic forms over $F^2$.

(Munemasa-Pasechnik-Shpectorov, 1993)

By Kabanov-Makhnev-Paduchikh (2007), we must have $q = 2^n$. 

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A graph $\Gamma$ is locally rank 3 with respect to $G \leq \text{Aut}(\Gamma)$ if and only if $G$ is listed here:

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$n$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}Q_n$</td>
<td>$n \geq 5$</td>
<td>$2^{n-1} \rtimes S_n$, $2^{n-1} \rtimes A_n$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$A_4$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$2^4 \rtimes S_4$, $2^3 \rtimes S_4$, $(2^3 \rtimes A_4).2$</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>$2^8 \rtimes \text{PGL}_2(8)$</td>
</tr>
<tr>
<td></td>
<td>11, 12, 23, 24</td>
<td>$2^{n-1} \rtimes M_n$</td>
</tr>
<tr>
<td>$\frac{1}{2}\square_n$</td>
<td>$n \geq 8$ even</td>
<td>$2^{n-2} \rtimes S_n$, $2^{n-2} \rtimes A_n$</td>
</tr>
<tr>
<td></td>
<td>12, 24</td>
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</tr>
<tr>
<td>$\frac{1}{2}\Gamma(C_{23}).2$</td>
<td>23</td>
<td>$2^{11} \rtimes M_{23}$</td>
</tr>
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